MATHEMATICS

# HOMOGENEOUS MANIFOLDS ALL OF WHOSE GEODESICS ARE CLOSED

BY

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*Introduction*

Let us consider a connected Riemannian manifold $M$ which has the property that all its geodesics starting at a point $p$ in $M$ are simply closed of length $L$. $M$ is necessarily compact. R. BOTT [4] has determined the cohomology of manifolds of this type by making use of the theory of M. Morse. In this paper we make the additional hypothesis that $M$ is *homogeneous*, and show that then $M$ is homeomorphic to a symmetric space of rank one, at least in the even-dimensional case. The proof is purely topological, the geometry having been already exploited to yield Bott's theorem.

This paper is a natural continuation of [5], to which we shall refer frequently for various results needed here.

The principal results of the present paper are stated in Theorem I and Theorem II.

1. *The topological problem*

Let $M$ be a manifold having the property stated in the introduction. BOTT's theorem [4] states that $\pi_1(M)=0$ or $\mathbf{Z}_2$ and that $H^*(M;\mathbf{Z})$ is a truncated polynomial algebra in the first case, while the universal covering $\tilde{M}$ of $M$ is an integral homology sphere in the second case.

We shall consider here *simply connected* manifolds $M$ of *even* dimension. $H^*(M;\mathbf{Z})$ is then a truncated polynomial algebra on one generator $\gamma$ of even dimension.

Now recall the following

Theorem. (J. F. ADAMS [1]) If for the finite polyhedron $X$, $H^*(X;\mathbf{Z}_2)$ is a truncated polynomial algebra generated by one element $\gamma$, then either $\gamma^2=0$, or $\gamma^2\neq 0$ and then dim $\gamma=1$, 2, 4, or 8. Moreover if dim $\gamma=8$, then $\gamma^3=0$. (This last result is due to J. ADEM.)

Applying this result to our situation, we find that either dim $\gamma=$ dim $M$ and $\gamma^2=0$, or dim $\gamma=2$, 4, or 8, and $\gamma^2\neq 0$.

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Let us now assume moreover that $M$ is a *homogeneous* Riemannian manifold. Our problem is to classify all homogeneous manifolds whose integral cohomology is of the type described above.

## 2. *Homogeneous manifolds of even dimension, whose integral cohomology algebra is of the form $Z[x]/(x^k)$*

Throughout this section we shall let $M$ stand for a *simply connected* space of this type.

Lemma 2.1. $M$ has positive Euler characteristic $\chi$, and we may write $M=G/K$ where $G$ is a compact, connected simple Lie group, and $K$ is a connected subgroup of maximal rank in $G$.

Proof: The first statement is immediate. Since $M$ is connected, the identity component of the Lie group of all isometries acts transitively. So $M=G/K$ with $G$ connected, and $K$ is connected of maximal rank because $\chi(M)>0$ and $\pi_1(M)=0$. We still have to show that $G$ is simple. Now $G$ is centerless [5; Lemma 1.1] and so $G=G_1\times G_2\times\ldots\times G_r$, with $G_i$ compact, connected and simple. Since $K$ has maximal rank, we have that $K=K_1\times K_2\times\ldots\times K_r$, where $K_i\subset G_i$, $i=1,\ldots,r$. [5; Lemma 1.2]. It follows that $M=\prod_1^r (G_i/K_i)$. But the cohomology algebra of a product of two or more factors can not be generated by a single element. Q.E.D.

We now distinguish several cases.

*Case I.* $\dim\gamma=\dim M$; $\gamma^2=0$. We have $\chi(M)=2$. It follows that $K$ is a maximal connected subgroup of maximal rank. (Lemma 1.3 in [5]).

*Case II.* $\dim\gamma=8$; $\gamma^2\neq 0$; $\gamma^3=0$. We have $\dim M=16$ and $\chi(M)=3$. Again $K$ is maximal connected of maximal rank.

*Case III.* $\dim\gamma=2$; $\gamma^2\neq 0$, so $\dim M\geqslant 4$. It is convenient to represent $M$ as $G/K$ where now $G$ is compact, simple and *simply connected*, and $K$ is connected of maximal rank in $G$. The case at hand needs special care. First we prove

Lemma 2.2. $K$ is locally of the form $T_1\times K_{ss}$ where $T_1$ is a one-dimensional torus and $K_{ss}$ is the semi-simple part of $K$.

Proof: Consider the homotopy sequence of the fibration $G\to M$ with fiber $K$:

$$\to\pi_2(G)\to\pi_2(M)\to\pi_1(K)\to\pi_1(G)\to\pi_1(M)\to.$$

We know that $\pi_1(G)=\pi_2(G)=0$, so $\pi_1(K)\approx\pi_2(M)\approx\mathbf{Z}$ by the Hurewicz theorem. Let $Z_0(K)$ be the connected 1 component of the center of $K$. One knows that $K$ is homeomorphic to $Z_0(K)\times K_{ss}$. Therefore $\pi_1(K_{ss})=0$, $\pi_1(Z_0(K))\approx\mathbf{Z}$ and $Z_0(K)$ is a circle. Q.E.D.

Note that as far as topology is concerned $K$ may be regarded as a product $T_1\times K_{ss}$, where $T_1$ is a one-dimensional torus.

Lemma 2.3. $K_{ss}$ is simple.

Proof: Consider, for the *real* cohomology, the spectral sequence of the fibration $G \to M$. If $\beta$ is a non-zero element of $H^2(M;\mathbf{R})$, then $\beta^k$ is a generator of $H^{2k}(M;\mathbf{R})$ for $k=1, 2, \ldots$ dim $M/2$. We have that $H^1(K) \approx \mathbf{R}$. Let $\alpha'$ be a generator. Then $E_2^{0,1} \approx H^0(M) \otimes H^1(K)$ is one-dimensional with generator $1 \otimes \alpha' = \alpha$ say. $E_3^{0,1} = \operatorname{Ker}(d_2\colon E_2^{0,1} \to E_2^{2,0}) = E_\infty^{0,1} = 0$ since $H^1(G)=0$. It follows that $d_2\colon E_2^{0,1} \to E_2^{2,0}$ is an isomorphism; therefore $d_2\alpha = \gamma \otimes 1 \neq 0$ is a generator of $E_2^{2,0} = H^2(M) \otimes H^0(K) \approx \mathbf{R}$. Now $E_2^{2,1} \approx H^2(M) \otimes H^1(K)$ with generator $\gamma \otimes \alpha'$. Then $d_2(\gamma \otimes \alpha') = d_2 d_2\alpha \cdot \alpha + (d_2\alpha)(d_2\alpha) = (\gamma \otimes 1)(\gamma \otimes 1) = \gamma^2 \otimes 1$. So $d_2\colon E_2^{2,1} \to E_2^{4,0}$ is also an isomorphism. Hence $E_3^{4,0} = E_4^{4,0} = 0$. We now look at $E_2^{0,3} \approx H^0(M) \otimes H^3(K) \approx H^3(K)$. Since $E_2^{2,2}=0$, we have $E_3^{0,3} = E_2^{0,3}$. Since $E_3^{3,1}=0$, $E_4^{0,3} = E_3^{0,3}$ and $E_\infty^{0,3} = E_5^{0,3} = \operatorname{Ker}(d_4\colon E_4^{0,3} \to E_4^{4,0})$. Hence $E_\infty^{0,3} = E_4^{0,3}$ because $E_4^{4,0} = 0$ as shown.

Now $H^3(G)$ is one-dimensional, since $G$ is simple. Therefore dim $E_\infty^{0,3} \leqslant 1$, and thus also dim $H^3(K) \leqslant 1$. If dim $H^3(K)=0$, then $K$ and $G$ both must have rank 1, so $G \approx SU(2)$, which is 3-dimensional. It follows that dim $M<4$, a contradiction. Hence the third Betti-number of $K$ is equal to 1, which implies that $K_{ss}$ is simple. Q.E.D.

We now turn our attention to the next

*Case IV*. dim $\gamma = 4$; $\gamma^2 \neq 0$, dim $M \geqslant 8$. In this case we have

Lemma 2.4. If we set $M = G/K$, with $G$ compact, simple and simply connected, then $K$ is the direct product of two compact and connected simple groups.

Proof: By the Hurewicz theorem we have $\pi_4(M) \approx H_4(M) \approx \mathbf{Z}$. It is clear that $\pi_1(K)=0$, so $K$ is semi-simple, and is the direct product of compact simple groups. Consider now the following portion of the homotopy sequence of the fibration $G \to M$:

$$\pi_4(G) \to \pi_4(M) \to \pi_3(K) \to \pi_3(G) \to \pi_3(M).$$

Or,

$$\pi_4(G) \to \mathbf{Z} \to \pi_3(K) \to \mathbf{Z} \to 0.$$

Now $\pi_4(G)$ is finite, so the first homomorphism is zero, and hence $\pi_3(K) \approx \mathbf{Z} \oplus \mathbf{Z}$, and $K$ is indeed the direct product of *two* simple groups. Q.E.D.

We now discuss the cases III and IV simultaneously. First we need

Lemma 2.5. Let $G$ be a compact, connected Lie group, $K$ a subgroup of maximal rank such that $H^*(G/K;\mathbf{R})$ is generated by a single element. Let $U$ be a maximal connected subgroup containing $K$. Then $H^k(G/U;\mathbf{R})$ is at most one-dimensional, $k=0, 1, \ldots$.

Proof: Consider the spectral sequence, for real cohomology, of the fibering $p: G/K \to G/U$, with fiber $U/K$. Since the Betti numbers in odd dimensions of $G/U$ and $U/K$ vanish, the spectral sequence is trivial, and so $p^*: H^*(G/U; \mathbf{R}) \to H^*(G/K; \mathbf{R})$ is injective. Q.E.D.

For brevity we shall say that a space $X$ has *property* $(R)$ if $H^k(X; \mathbf{R}$ is at most one-dimensional ($k=0, 1, \ldots$), and that it has *property* $(Z)$ if $H^*(X; \mathbf{Z})$ is generated by a single element.

Now we first examine whether an *exceptional* Lie group may contain a *non-maximal* subgroup of maximal rank such that the corresponding homogeneous space has property $(Z)$. In table I we have listed all possible maximal subgroups of maximal rank which could contain a proper subgroup of the type wanted. We use here the table given in [3], and also lemma 2.3 and lemma 2.4. We also give the relevant parts of the Poincaré-polynomials over $\mathbf{R}$ of the corresponding homogeneous spaces.

TABLE I

| $K$ | $K$ | $P(G/K, t)$ |
|---|---|---|
| $E_6$ | $A_1 \times A_5$ | $1+t^4+t^6+t^8+2t^{12}+\ldots$ |
| $E_7$ | $A_1 \times D_6$ | $1+t^4+2t^8+\ldots$ |
| | $A_7$ | $1+t^6+ \quad +2t^{18}+\ldots$ |
| $E_8$ | $D_8$ | $1+t^8+t^{12}+2t^{16}+\ldots$ |
| | $A_1 \times E_7$ | $1+t^4+t^8+2t^{12}+\ldots$ |
| | $A_8$ | $1+t^6+t^8+t^{10}+2t^{12}+\ldots$ |
| $F_4$ | $A_1 \times C_3$ | $1+t^4+2t^8+\ldots$ |
| | $B_4$ | $1+t^8+t^{16}$ |
| $G_2$ | $A_2$ | $1+t^6$ |
| | $A_1 \times A_1$ | $1+t^4+t^8$ |

From Lemma 2.5 and the information of table I we deduce that the only possibilities for a non-maximal $K$ in exceptional $G$ are (we are still investigating cases III and IV): $G=F_4$, $K=T \times B_3$, $T \times A_3$ or $B_1 \times D_3$. It is easy to verify that the corresponding homogeneous spaces do not have property $(R)$. $G=G_2$, $K=T \times A_1 \subset A_2$ or $K=T \times A_1 \subset A_1 \times A_1$. We shall show that the corresponding homogeneous spaces do not have property $(Z)$. There are two cases to consider:

(i) $G_2/T \times A_1$ where $T \times A_1 \subset A_2$. The homogeneous space is the manifold of oriented 2-planes in $\mathbf{R}^7$. It is known, more generally, that $H^*(SO(n+2)/SO(2) \times SO(n), \mathbf{Z})$ is *not* generated by a single element. See [2].

(ii) $G_2/T \times A_1$ where $T \times A_1 \subset A_1 \times A_1$. It is known that the subgroup of type $A_1 \times A_1$ in $G_2$ is actually isomorphic to $SO(4)=SO(3) \times SU(2)$. The subgroup $T \times A_1$ we are considering is then $T \times SU(2) \subset SO(3) \times SU(2)$. Put $X = G_2/T \times SU(2)$, and $B = G_2/SO(4)$. From general facts it is clear that $X$ has no torsion and that the additive structure of $H^*(X; \mathbf{Z})$ is given by: $H^{2k}(X; \mathbf{Z}) \approx \mathbf{Z}$ for $k = 0, 1, \ldots, 5$, and $H^q(X; \mathbf{Z}) = 0$ otherwise. We shall consider the fibering $X \to B$ with fiber $SO(3)/T = S^2$, and work for the moment with coefficients in $\mathbf{Z}_2$. We need the following facts [2]: $P_2(B, t) = 1 + t^2 + t^3 + t^4 + t^5 + t^6 + t^8$. As an algebra, $H^*(B; \mathbf{Z}_2)$ has two generators of degrees 2 and 3 respectively, satisfying the relations $\alpha^3 = \beta^2$ and $\beta\alpha^2 = 0$, dim $\alpha = 2$, dim $\beta = 3$.

Let $\mu$ be a generator of $H^*(S^2, \mathbf{Z}_2)$. In the spectral sequence of the fibering $X \to B$ we have $E_2 = E_3$. The map $d_3 \colon E_3^{0,2} \to E_3^{3,0}$ is an isomorphism in view of the fact that $H^3(X; \mathbf{Z}_2) = 0$. So $d_3(1 \otimes \mu) = \beta \otimes 1$. Also we have that $E_\infty^{0,2} = E_4^{0,2} = 0$, and $E_2^{2,0} = E_3^{2,0} = E_\infty^{2,0}$. It follows that $H^2(X)$ may be identified with $E_3^{2,0} = H^2(B) \otimes H^0(S^2)$, which is one-dimensional with generator $\alpha \otimes 1$. Let $\lambda \in H^2(X)$ correspond to $\alpha \otimes 1$ under this identification. In fact $\lambda$ is in the subgroup $D^{2,0}$ of the filtration: $H^2(X; \mathbf{Z}_2) = D^{0,2} \supset D^{1,1} \supset D^{2,0} \supset 0$, and $D^{2,0} = E_\infty^{2,0} = E_3^{2,0}$. We shall show that $\lambda^3 = 0$. For this it is sufficient to show that $E_\infty^{6,0} = D^{6,0} = 0$. Consider $d_3 \colon E_3^{3,2} \to E_3^{6,0}$. Now a generator of $E_3^{3,2}$ is $\beta \otimes \mu = (\beta \otimes 1)(1 \otimes \mu)$; so $d_3(\beta \otimes \mu) = \beta^2 \otimes 1 = \alpha^3 \otimes 1 \neq 0$. Again then $d_3 \colon E_3^{3,2} \to E_3^{6,0}$ is surjective, so that $E_4^{6,0} = 0$, and hence also $E_\infty^{6,0} = 0$.

It follows that none of the homogeneous spaces of $G_2$ has integral cohomology of the desired structure.

The results obtained so far allow us to state: for $G$ exceptional there can be no non-maximal $K$ of maximal rank such that the space $G/K$ has property $(Z)$.

We now turn to the classical groups. In table II we have listed the possible non-maximal subgroups for the cases III and IV.

TABLE II

| $G$ | $K$ | $P(G/K, t)$ |
|---|---|---|
| $B_n$ | $T \times A_{n-1}$ | $(1+t^2)(1+t^4) \ldots \ldots (1+t^{2n})$ |
| | $T \times D_{n-1}$ | $(1+t^{2n-2})(1+t^2+ \ldots \ldots +t^{4n-2})$ |
| | $D_k \times D_{n-k}$ | |
| $C_n$ | $T \times C_{n-1}$ | $1+t^2+t^4+ \ldots \ldots +t^{4n-2}$ |

The homogeneous space corresponding to $(C_n, T \times C_{n-1})$ is a complex projective space, and in particular it is a symmetric space. From the Poincaré-polynomials given in Table II it is seen that only the case $(B_n, D_k \times D_{n-k})$ needs further consideration. We dispose of the corresponding homogeneous space $X$ by the following calculation; we are interested in the case where dim $X$ is divisible by 4; then $n$ is even. Also we may suppose that $k \geqslant 3$, $n-k \geqslant 3$ (since $D_2$ is not simple), and hence that $n \geqslant 6$. If we put $n=2p$, we have $p \geqslant 3$. Now

$$\chi(X) = (2^n \cdot n!)/2^{k-1} \cdot k! \cdot 2^{n-k-1} \cdot (n-k)! = 4\binom{n}{k},$$

and dim $X = 2n + 4k(n-k)$. Hence dim $X < \chi(X)$, which shows that $H^*(X;\mathbf{R})$ can not have a single generator in dimension 4.

We have proved

Proposition 2.1. Let $G$ be a simple, compact and connected Lie group, $K$ a connected non-maximal subgroup of maximal rank. Then $G/K$ has property ($Z$) only if $G = C_n$ and $K = T \times C_{n-1}$, so that $G/K$ is a complex projective space.

It is now easy to prove our main result:

Theorem I. Let $M$ be a simply connected Riemannian homogeneous manifold of even dimension, having the property that all its geodesics are simply closed of length $L$. Then $M$ is homeomorphic to a symmetric space.

Proof: If we put $M = G/K$, then $K$ is maximal connected, except for the one case of Proposition 2.1, which however yields a symmetric space. For maximal $K$ the result follows from the proof of Lemma 3.2 in [5]. Q.E.D.

As another application of our methods we determine the compact Lie groups that can act transitively on certain projective spaces. It should be noted that a manifold $M$ as in Theorem I must be of rank one (as a symmetric space), and so is homeomorphic to a sphere, a complex projective space, a quaternion projective space, or the Cayley plane.

Let us denote by $P(\mathbf{C}^n)$ the complex projective space of real dimension $2(n-1)$, $n > 2$; by $P(\mathbf{H}^n)$ the quaternion projective space of real dimension $4(n-1)$, $n > 2$; and by $P(\mathbf{Cay})$ the Cayley plane. Furthermore let $PSU(n)$ (resp. $PSp(n)$) stand for the complex (resp. quaternionic) projective group.

Our results can be used to deduce the following

Theorem II. Let $G$ be a compact connected Lie group acting transitively and effectively on one of the spaces $P(\mathbf{C}^n)$, $P(\mathbf{H}^n)$ or $P(\mathbf{Cay})$. Then $G$ must be as follows:

for $P(\mathbf{C}^n)$, $G = PSU(n)$, $n > 2$ or $G = PSp(m)$ if $n = 2m$, $m > 1$;

for $P(\mathbf{H}^n)$, $G = PSp(n)$, $n > 2$;

for $P(\mathbf{Cay})$, $G = F_4$.

Proof: Let $M$ stand for such a projective space. Let us represent $M$ as $M = G/K$, where $G$ is compact, simple and simply connected. It is enough to find $G$. From Proposition 2.1 we see that we need only look at pairs $(G, K)$ where $K$ is maximal connected. The cases that have not yet been excluded by Table I or Lemma 3.2 of [5] are irreducible symmetric spaces of type (in E. Cartan's notation) *AIII*, *BDI*, *CI*, *CII*, *DIII*, *EII*, *EIII*, *EVI*, *EIX*, *FI*, *FII*, and $G$. Taking into account the isomorphisms that exist (in low dimensions) between the irreducible symmetric spaces of different types, we see that the various projective spaces occur only when $G$ corresponds to the goups mentioned in Theorem II. Q.E.D.

Remarks.

1. We have considered in this paper only simply connected manifolds of even dimension. The restriction to simply connected spaces is not serious. The reason why we have excluded odd-dimensional manifolds is the following: if $m = \dim M$ is odd, (suppose $m > 3$) then $M$ is an integral homology sphere by Bott's Theorem. By Theorems of Hurewicz and Whitehead, $M$ is then of the homotopy type of an $m$-sphere, and hence by Smale's Theorem ("Generalized Poincaré's conjecture in dimensions greater than four", Ann. of Math. (2) **74**, 391–406) $M$ is a topological sphere.

2. After completion of the present investigation we learned of a paper (in Russian) by A. L. Oniščik which also contains Theorem II. See Math. Rev. **27**, (1964), No. 5868.

3. New examples of spaces whose integral cohomology algebra is a truncated polynomial algebra, were discovered by J. Eells and N. H. Kuiper. ("On manifolds which are like projective planes", Publ. Math. I.H.E.S. No. **14**.)

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